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Polygonal Billiards with Small Obstacles

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The technique of *unfolding* a polygonal billiard table is used to answer certain questions concerning the illumination problem. The main problem addressed is how many point obstacles would suffice to block any billiard path between two points of the polygon. The answer can then be generalized from point obstacles to small ε-neighborhoods of points.

KEY WORDS AND PHRASES: Billiards; dynamical systems; symmetry groups; transitivity; polygon.

INTRODUCTION AND DEFINITIONS

This paper was inspired by a recent article[1] giving an example of a polygonal billiard, such that some pairs of points inside the billiard table cannot be connected by a light ray (or, equivalently by a billiard trajectory) which is reflected at the edges. (See e.g., refs. 2, 3, and 6). The proofs given in ref. 1 rest on the convention that any billiard path entering a corner stops there. This may be regarded as somewhat unsatisfactory and raised objections. An alternative would be took use the following convention (see Fig. 1 for an illustration): If a path hits a corner, consider a sequences of points on each of the two edges forming the corner such that the points in each sequence converge to the corner. For each point in one of the two sequences lying close to the corner consider a straight line from the last reflection of the original path to the point in question. Since these paths hit an edge their reflected paths are well-defined. As the sequences of

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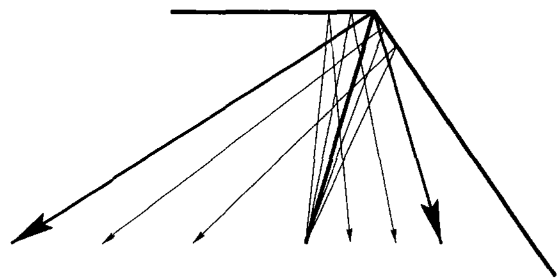

Fig. 1. Definition of the continuations of a path hitting a corner.

points converge to the corner, the paths associated with them converge to the original path prior to hitting the corner. The two sequences of reflected paths converge to two paths which we will define to be the continuations of the path hitting a corner. Note that in the case that the angle in the corner is of the form $360°/n$ for some $n \in \mathbb{N}$ the two continuations agree.

The question addressed is as follows: Given a pair of points, L and A, in a plane polygon $\mathscr{P}$, is it possible to find a finite configuration $P_1,..., P_s$ of points in $\mathscr{P}$ such that any trajectory from L to A passes through one of these points before reaching A? In other words, putting obstacles at $P_1,..., P_s$ would make it impossible for a light source at L to lighten point A. Such a configuration is called *blocking*, and a polygon is said to have the *finite blocking property* if a blocking configuration exists for any pair L, $A \in \mathscr{P}$ and the number of obstacles is uniformly bounded. The motivation for such a consideration is that we want to be able to assess the number of ball (in two dimensions, disks) as obstacles preventing two billiard balls from hitting one another, uniformly with respect to the radii of the billiard balls and the size of the billiard table. We comment on this aspect and give the definition of the blocking property for this case in Section 4.

2. OBSTACLES IN THE SQUARE

As an illustratory example we will take the case of the square $\mathscr{S} = [0, 1]^2$. First, consider the case where L is at the origin $O = (0, 0)$.

Theorem 1. For any point $A \in \mathscr{S}$ there exists a blocking configuration containing at most four points.

Proof. Using the technique of *unfolding* of trajectories (see e.g., refs. 4 and 5) we can transform each billiard trajectory in the square into a halfline in the plane starting at the origin (see Fig. 2): Note that the unfolding of

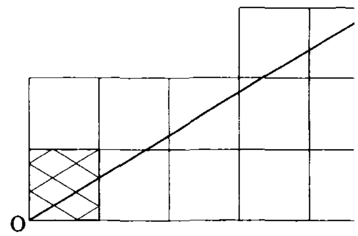

Fig. 2. Unfolding of a path into a straight line.

paths respects the convention on paths hitting corners. The mirror-images of the point $A = (x, y)$, denoted by $\mathcal{M}(A)$, are:

$$\left.\begin{array}{c}(x, y)\\(x, 2-y)\\(2-x, y)\\(2-x, 2-y)\end{array}\right\} + (2m, 2n) \tag{1}$$

We show that the following set of 4 points is a blocking configuration:

$$\left(\frac{x}{2}, \frac{y}{2}\right), \left(\frac{x}{2}, 1-\frac{y}{2}\right), \left(1-\frac{x}{2}, \frac{y}{2}\right), \left(1-\frac{x}{2}, 1-\frac{y}{2}\right) \tag{2}$$

See Fig. 3 for an illustration of the blocking set. Using the symmetry in this definition we get that the mirror-images of this set are:

$$\left.\begin{array}{c}\left(\frac{x}{2}, \frac{y}{2}\right)\\ \left(\frac{x}{2}, 1-\frac{y}{2}\right)\\ \left(1-\frac{x}{2}, \frac{y}{2}\right)\\ \left(1-\frac{x}{2}, 1-\frac{y}{2}\right)\end{array}\right\} + (m, n). \tag{3}$$

Consider now a path from O to a mirror-image A' of A. We will show that its half-way point is a mirror-image of an obstacle.

Case 1. $A' = (x, y) + (2m, 2n)$. The halfway-point is $(x/2 + m, y/2 + n)$, which is included in (3) and so is a mirror-image of an obstacle.

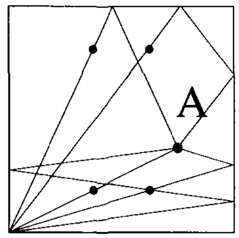

Fig. 3. The obstacles for the point A.

Other cases. These are analogous. ∎

Remark. If L is an arbitrary point of the square a similar construction gives us a blocking set of at most 16 points. The proof is analogous to the one above, but will not be given in this section, as the result follows as a corollary from a more general statement in Section 3.3.

3. GENERAL CASE

3.1. Group Structure

We introduce the group $\mathbf{G}_{\mathscr{P}}$ of motions of $\mathbb{R}^2$ related to an n-edge polygon $\mathscr{P}$. $\mathbf{G}_{\mathscr{P}}$ is generated by the reflections of $\mathbb{R}^2$ at the edges of the polygon in the following way: First one labels the edges $a_1,..., a_n$, counterclockwise, say. Then any reflection process can be denoted by the word corresponding to the sequence of edges at which the polygon is reflected. In this process we reflect at the *current* position of an edge of the polygon, i.e., its position in the plane after the previous reflections. The group operation is concatenation of words. The relations are the sequences of reflections that act on $\mathbb{R}^2$ as the identity map. So e.g., one set of relations is of the form a_i^2. As we concatenate at the right and of a word, the reflections are to be read from left to right. $\mathbf{G}_{\mathscr{P}}$ is a quotient of the free group on n generators. If it is clear which is the underlying polygon we may omit the subscript $\mathscr{P}$.

3.1.1. Examples. *The Square.* For the square $\mathscr{S}$, $\mathbf{G}_{\mathscr{S}}$ is generated by $a_1,..., a_4$. In addition to the relations a_i^2 we have the relations $(a_i a_{i+1})^2$ and the reverse relations $(a_i a_{i-1})^2$, where $a_5 = a_1$ and $a_0 = a_4$. Thus any mirror-image A' of a point A in $\mathscr{S}$ can be written in the following way:

$$A' = (a_1^{\varepsilon} a_2^{\mu} (a_1 a_3)^m (a_2 a_4)^n) A$$

where $\varepsilon, \mu \in \{0, 1\}$, $m, n \in \mathbb{Z}$. Note that $a_1 a_3$ is a translation by $(-2, 0)$ and $a_2 a_4$ is a translation by $(0, 2)$ in the (x, y)-plane.

Equilateral Triangle. For the equilateral triangle $\mathscr{T}$, we have three edges a_1, a_2, a_3 and the group $\mathbf{G}_{\mathscr{T}}$ is given by:

$$\langle a_1, \ldots, a_3 \mid a_i^2, 1 \leqslant i \leqslant 3; (a_i a_j)^3, i \neq j \rangle$$

3.2. THE GROUP ACTION ON $\mathbb{R}^2$

Any element of $\mathbf{G}_{\mathscr{P}}$ is an isometry of the plane and can therefore be written as a product of a translation, a rotation and, possibly, a reflection. We will in general denote reflections by R, rotations by r and translations by t. (Remember that the way $\mathbf{G}$ is defined, a reflection a_j also reflects the reflection axis for the next reflection!) Consider the action of two successive reflections. We have two cases:

The two edges a_i and a_j are parallel: In this case, $a_i a_j$ is a translation by a vector orthogonal to a_i whose length is twice the distance from a_i to a_j.

The angle α from a_i to a_j is not zero: In this case, $a_i a_j$ is a rotation about the point of intersection of the two lines extending a_i and a_j with angle of rotation -2α. So, unless $\alpha = \pi/2$, a_i and a_j do not commute.

Now consider two successive rotations: Let r_1 and r_2 be two rotations with centres O_i and angles α_i. We have:

$$r_1 r_2 = r' t' \tag{4}$$

where r' has centre $r_2(O_1)$ and angle $\alpha_1 + \alpha_2$ and t' is the translation by vector $r_2(O_1) - O_1$. In general, two rotations do not commute.

Next we consider products of translations, reflections and rotations. Let f and g be two transformations of these types. Define ${}^g[h]$ be the transformation of the same type as h, where the parameters of h, e.g., the vector of translation or the axis of reflection or the centre and angle of rotation, have been transformed by g first. We have:

Lemma 2. We have the following relations:

$$Rt = {}^R[t]\, R \tag{5}$$

$$rt = {}^r[t]\, r \tag{6}$$

$$Rr = {}^R[r]\, R \tag{7}$$

Proof. By geometric inspection. ∎

Define $\mathbf{T}_{\mathscr{P}}$ to be the subgroup of $\mathbf{G}_{\mathscr{P}}$ consisting of translations.

Lemma 3. The translations form a normal subgroup.

Proof. We have to show that

$$g^{-1}\mathbf{T}g \subseteq \mathbf{T}$$

for all $g \in \mathbf{G}$. We have by the statements (5) and (6) of Lemma 2 that for any translation $t \in \mathbf{T}$, $g^{-1}tg = {}^{g^{-1}}[t]\ g^{-1}g$ is also a translation in $\mathbf{G}$ and hence $g^{-1}\mathbf{T}g \subseteq \mathbf{T}$. ∎

So we can now define the quotient

$$\mathbf{H}_{\mathscr{P}} := {}^{\mathbf{H}_{\mathscr{P}}}/_{\mathbf{T}_{\mathscr{P}}}$$

Every element g of $\mathbf{G}_{\mathscr{P}}$ can be written as

$$g = ht \tag{8}$$

where $h \in \mathbf{G}$ is a representative of a coset of $\mathbf{T}$ and $t \in \mathbf{T}$. We can always choose h uniquely so that it does not contain a translation. Next we look at the set $\mathscr{M}(A) = \{g(A) \mid g \in \mathbf{G}_{\mathscr{P}}\}$, the orbit of A under $\mathbf{G}_{\mathscr{P}}$.

Theorem 4.

$$\mathscr{M}(A) = \bigcup_i \left(\left(\sum_j \mathbb{Z}v_j\right) + B_i\right) \tag{9}$$

where $\{B_i\}$ are the images of A under representatives of $\mathbf{H}_{\mathscr{P}}$ and the v_j's are the generators of $\mathbf{T}_{\mathscr{P}}$.

Proof. This follows from Eq. (8): Let $g(A)$ be a mirror-image of A. Then $g(A) = (ht)(A)$, therefore $\exists h' \in \mathbf{H}_{\mathscr{P}}$ such that $g(A) = h'(A) + \sum_{j=1}^{k} \lambda_j v_j$. Conversely, let $A' = h(A) + \sum_{j=1}^{k} \lambda_j v_j$. Then there exists $g \in \mathbf{G}_{\mathscr{P}}$ such that $A' = g(A)$. ∎

3.3. A Sufficient Condition

One of the main objectives is to determine which polygons possess the blocking property. A useful sufficient condition is given by

Theorem 5. The following is a sufficient condition for a polygon $\mathscr{P}$ to have the finite blocking property:

There exist finitely many vectors v_j, $1 \leqslant j \leqslant N$, and for any point $A \in \mathscr{P}$ there exist finitely many points A_i, $1 \leqslant i \leqslant k$, such that $\mathscr{M}(A)$ consists precisely of the points of the form:

$$A_i + \sum_{j=1}^{N} \lambda_j v_j \tag{10}$$

where $\lambda_j \in \mathbb{Z}$.

Proof. First we note that the convention on paths hitting corners is consistent with the unfolding principle; all paths from L to A can be expressed as straight lines from L to a mirror-image of A. So we obtain: In order to block the point A, it is sufficient to place obstacles at the half-way point on every path from L to a mirror-image of A. These half-way points have the following form:

$$\frac{1}{2}\left(L + A_i + \sum_{j=1}^{N} \lambda_j v_j\right)$$

This can be written as

$$P_{i,\lambda} + \sum_{j=1}^{N} \left\lfloor \frac{\lambda_j}{2} \right\rfloor v_j$$

where

$$P_{i,\lambda} = \frac{1}{2}(L + A_i) + \sum_{j=1}^{N} \left\{\frac{\lambda_j}{2}\right\} v_j \tag{11}$$

where $\lfloor \cdot \rfloor$ denotes the integer part, $\{x\} = x - \lfloor x \rfloor$, and $\lambda := (\lambda_1, \ldots, \lambda_N)$. Note that there are only finitely many $P_{i,\lambda}$. We want to take pre-images of the $P_{i,\lambda}$'s in the original polygon. For each $P_{i,\lambda}$ there is such a pre-image, as follows from:

Lemma 6. For every point $A \in \mathbb{R}^2$ there is a point $A' \in \mathscr{P}$ and $g \in \mathbf{G}_{\mathscr{P}}$ such that $g(A') = A$.

The proof of this lemma will be given at the end of the proof. So let $\{P'_{i,\lambda}\}$ be a set of pre-images in the original polygon for the $P_{i,\lambda}$'s. We will show that the $P'_{i,\lambda}$'s form a blocking set. For this it is enough to show that all points of the form (11) are contained in $\mathscr{M}(\{P'_{i,\lambda}\}) := \{g(\{P'_{i,\lambda}\}) \mid g \in \mathbf{G}_{\mathscr{P}}\}$.

By (10) we have that the mirror-images of $P'_{i,\lambda}$ are precisely the points of form:

$$A_{i,\lambda,l}+\sum_{r=1}^{N}\lambda_{i,j,r}v_r \tag{12}$$

where l runs through a finite set and $\lambda_{i,j,r}$ runs through $\mathbb{Z}$.

As the $P'_{i,\lambda}$'s are pre-images of the $P_{i,\lambda}$'s, we know that the $P_{i,\lambda}$'s are contained in (12). Therefore, $\forall i, \lambda \exists l', \lambda'_{i,j,r}$ such that

$$P_{i,\lambda}=A_{i,\lambda,l'}+\sum_{r=1}^{N}\lambda'_{i,j,r}v_r$$

But then all points of the form

$$A_{i,\lambda,l}+\sum_{r=1}^{N}\lambda_{i,j,r}v_r$$

where the λ_r's run through $\mathbb{Z}$, are also in $\mathcal{M}(\{P'_{i,\lambda}\})$ by (12). So all half-way points are in $\mathcal{M}(\{P'_{i,\lambda}\})$, as they are all of the form

$$P_{i,\lambda}+\sum_{r=1}^{N}\lambda_r v_r$$

with $\lambda_r \in \mathbb{Z}$, and we have arrived at a finite set of obstacles which form a blocking set. ▮

Proof of Lemma 6. Assume false. Then there is a point $A \in \mathbb{R}^2$ that does not have a pre-image in $\mathcal{P}$. Pick an arbitrary point X in $\mathcal{P}$ and connect it via a straight line to A. Move along this line and reflect the polygon whenever its hits an edge (or a mirror-image of an edge). If this line hits a vertex (or a mirror-image of a vertex), which may happen only after a finite number of reflections, choose another starting point X. As there are only countably many mirror-images of vertices, there is always a suitable starting point. Once we have reached A, we have that there is a mirror-image of $\mathcal{P}$ that contains A. But that means that there is a pre-image A' of A in $\mathcal{P}$. ▮

Remarks. The proof provides an upper bound for the number of obstacles needed. From (11) we see that there is a blocking set with $k2^N$ obstacles (for every A_i and for every v_j there are two cases: $\lambda_j \equiv 0$ or $1 \pmod 2$)).

The proof is non-constructive, as the proof of Lemma 6 does not specify A', but under the additional condition on the mirror-images of $\mathscr{P}$:

The number of mirror-images of $\mathscr{P}$ lying above any point of $\mathbb{R}^2$ is bounded by a constant l.

We obtain a "constructive" version by taking all pre-images given by the reflection process instead of using Lemma 6 to obtain a pre-image in the original polygon. For the advantage of having a finite algorithm giving us a blocking set we have to pay the price of increasing the bound by a factor of l.

For the unit square the points A_j for the point $A(x, y)$ are (x, y), $(2-x, y)$, $(x, 2-y)$ and $(2-x, 2-y)$ and the vectors v_j are $(2, 0)$ and $(0, 2)$, so that we obtain a blocking set of at most 16 points.

For the equilateral triangle with corners $(0, 0)$, $(1, 0)$ and $(1/2, \sqrt{3}/2)$ the points A_i, for $A = (x, y)$ are:

$$(x, \pm y), \left(\pm 1 \mp \frac{2}{\sqrt{3}} y, \pm \frac{\sqrt{3}}{2}(1-x)\right)$$

and the vectors v_j are $(0, \sqrt{3})$ and $(3/2, \sqrt{3}/2)$ so that we obtain a blocking set of at most 32 points.

For the regular hexagon we could demonstrate directly that it satisfies the conditions of Theorem 5, but it is easier to use the following theorem and the result for equilateral triangles.

3.4. Construction

Theorem 7. A polygon $\mathscr{P}$ constructed by repeatedly reflecting a polygon $\mathscr{Q}$ at its edges finitely many times has the finite blocking property if $\mathscr{Q}$ has (see Fig. 4).

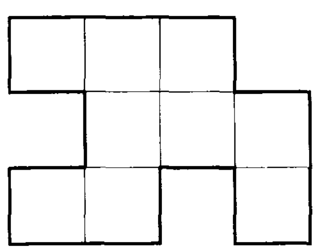

Fig. 4. This polygon has the blocking property.

Proof. Let A be the point in $\mathscr{P}$ we wish to block from a lightsource $L \in \mathscr{P}$. We can think of $\mathscr{Q}$ as the polygon containing L. Let A' be the pre-image of $A \in \mathscr{Q}$ under the process that constructed $\mathscr{P}$ from $\mathscr{Q}$. Now let $\{P_i\}$ be a finite set of obstacles needed to block A' from L in $\mathscr{Q}$. In order to block A from L in $\mathscr{P}$, we now take the following set of obstacles:

(a) the mirror-images of the P_i's obtained by the same reflecting process as the one used in constructing $\mathscr{P}$ from $\mathscr{Q}$,

(b) all points in $\mathscr{P}$ that are mirror-images of corners of $\mathscr{Q}$,

Now assume that there is a path in $\mathscr{P}$ from L to A. By following this path from A to L and reflecting the mirror-image of $\mathscr{Q}$ whenever the path goes through a mirror-image of an edge of $\mathscr{Q}$, we fold it into a path in $\mathscr{Q}$ that goes from L to A'. But as the P_i are a blocking set for A' in $\mathscr{Q}$, this is a contradiction. ∎

Remark. Because of the symmetry of the square this theorem together with Theorem 5 shows that every polygon constructed by glueing together squares (such that always vertices are glued onto vertices) has the finite blocking property (see Fig. 4).

3.5. Rational Polygons

Theorem 8. All rational polygons $\mathscr{P}$, (i.e., all polygons with π-rational angles) have the finite blocking property.

Proof. By Theorems 4 and 5 it is enough to show that

(a) $\mathbf{H}_{\mathscr{P}}$ is finite,

(b) $\mathbf{T}_{\mathscr{P}}$ is finitely generated.

(a) For a rational n-gon $\mathscr{P}$ all angles are of the from $m_i\pi/k$ for some $k \in \mathbb{N}$, $i = 1,..., n$. Using (4) and Lemma 2 we see that every word of length $\geqslant 2k$ contains a translation, because the angles of rotation are all of the form $2m_i\pi/k$. Hence $\mathbf{H}_{\mathscr{P}}$ contains at most n^{2k-1} elements.

(b) This follows from the fact that $\mathbf{H}_{\mathscr{P}}$ is finite. From (a) above we have that every word of length $\geqslant 2k$ contains a translation. Therefore there are less than n^{2k} ways to generate translations, as can be shown by the following argument: Suppose we have a translation $t = a_1 \cdots a_l$, where $l \geqslant 2k$. Then as $a_1 \cdots a_l$ contains at least one translation t_1 we can write $t = gt_1h$, Rewriting this we get:

$$t = gt_1h = g(t_1ht_1^{-1})\, t_1 = gh't_1$$

From the above we get that $gh' = tt_1^{-1} =: t_2 \in \mathbf{T}_{\mathscr{P}}$, so that $t = t_2 t_1$, i.e., t cannot be a generator. ∎

3.5.1. Regular *n*-gons. As regular n-gons $\mathscr{P}_n$ are rational, they possess the finite blocking property by the above theorem, which also gives us the following upper bound on the number of obstacles: The number of generators of $\mathbf{T}_{\mathscr{P}_n}$ is at most n^{2n-1}. For an upper bound on $|\mathbf{H}_{\mathscr{P}_n}|$ we have n^{2n-1}, as the total length of any word in $\mathbf{H}_{\mathscr{P}_n}$ is less than $2n$, so by the remark on p. 7 we have an upper bound of

$$n^{2n-1} 2^{n^{2n-1}}$$

A better upper bound can be obtained the following way: Split $\mathscr{P}_n$ into n congruent triangular "cake-slices." Let A_i be the points in the polygon obtained by taking the counterparts, in each triangle, of A and A', where A' is the image of A under the reflection of the triangle's symmetry axis. We end up we a set of maximally $2n$ points $A^{(i)}$ (see Fig. 5). In order to exploit this symmetry we use a generalization of Theorem 5 from single points to sets, the proof of which is analogous. We write $\mathcal{M}(\mathcal{A})$ for the orbit of the set $\mathcal{A}$ under $\mathbf{G}_{\mathscr{P}}$.

Theorem 9. The following is a sufficient condition for a polygon $\mathscr{P}$ to have the finite blocking property:

There exist finitely many vectors v_j, $1 \leqslant j \leqslant N$, and for any finite set of points $\mathcal{A} \in \mathscr{P}$ there exist finitely many points A_i, $1 \leqslant i \leqslant k$, such that $\mathcal{M}(\mathcal{A})$ consists precisely of the points of the form:

$$A_i + \sum_{j=1}^{N} \lambda_j v_j$$

where $\lambda_j \in \mathbb{Z}$.

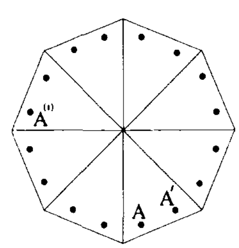

Fig. 5. The points $A^{(i)}$ in the case of an octagon.

As in the remark after Theorem 5 we have a blocking set of at most $k2^N$ points. We apply this to the two cases n odd and n even separately:

> n **odd**: In this case we see that the product of two reflections acts as a translation on the set $\mathscr{A} := \{A^{(i)}\}$, so that the conditions of Theorem 9 are satisfied with $k = 2n(n+1)$ (the original polygon plus n reflections) and $N = n(n-1)$. We obtain an upper bound
>
> $$2n(n+1)\,2^{n(n-1)}$$
>
> n **even**: In this case we have that every reflection acts as a translation on the set $\{A^{(i)}\}$, and that $N = n/2$. We obtain an upper bound
>
> $$2n2^{n/2}$$

4. APPLICATION TO BILLIARD BALLS OF POSITIVE SIZE

Though it is often convenient to represent physical objects such as lightsources or billiard balls by points this does not always capture all of the possible interactions between the original objects. In this section we generalise the results of the previous sections so that they apply to small balls as light sources, receptors and obstacles.

4.1. The Square

As an illustration we will again start witch the case of the square $\mathscr{S}$ with a point lightsource in the corner first. Instead of a single point we aim to block an ε-ball $B_\varepsilon(A) := \{x \in \mathscr{S} : |A - x| < \varepsilon\}$. From the proof of Theorem 1 we have that for a point $A = (x, y)$ a blocking set is given by Eq. (2). As the coordinates for the blocking points P_i depend linearly on the coordinates of A, we get that $B_\varepsilon(A)$ is blocked by four blocking balls $B_{\varepsilon/2}(P_i)$. Using the remark at the end of Section 2 about a general lightsource L in $\mathscr{S}$ we obtain that in the general case $B_\varepsilon(A)$ can be blocked by 16 balls $B_{\varepsilon/2}(P_i)$. Next we use the symmetry between L and A to obtain that all paths between two points of $B_\varepsilon(A)$ and $B_\varepsilon(L)$ respectively for two points A, L in $\mathscr{S}$ can be blocked by 16 obstacles $B_{2\varepsilon}(P_i)$.

4.2. Blocking of ϵ-balls in rational polygons

From Section 3.5 and the proof of Theorem 5 we get that, given a rational polygon $\mathscr{P}$ and points L and A in $\mathscr{P}$, there is a finite blocking set, whose elements P_i depend linearly on the coordinates of L and A. Therefore if we replace L and A by ε-balls we obtain that for any $\varepsilon > 0$ all paths between $B_\varepsilon(A)$ and $B_\varepsilon(L)$ are blocked by a finite number of balls $B_{2\varepsilon}(P_i)$.

Remark. Given two ε-balls $B_\varepsilon(A)$ and $B_\varepsilon(L)$, we can always find a finite number of blocking balls $B_{2\varepsilon}(P_i)$, by just simply building a "ring" around $B_\varepsilon(L)$, say. The special nature of the blocking configuration described in the preceding paragraphs is that the number of blocking balls does not depend on ε. Therefore we define:

> We say that a polygon has the *finite blocking property for disks* if there exists $n \in \mathbb{N}$ such that for all pairs of points L and A there exists a finite number of points P_i, $1 \leqslant i \leqslant n$ with the property that for all $\varepsilon > 0$ all paths between two points of $B_\varepsilon(A)$ and $B_\varepsilon(L)$ respectively are blocked by $\bigcup_{i=1}^{n} B_{2\varepsilon}(P_i)$.

Using this definition we obtain:

Theorem 10. All rational polygons possess the finite blocking property for disks. ∎

5. CONCLUDING REMARKS

We have seen that in a wide class of polygons it is possible to block all paths between two points by placing a finite number of obstacles into the polygon. There still remain open problems, some of which we have conjectures about:

We have not yet found a polygon that does *not* have the blocking property, although we conjecture that only the rational polygons do.

We conjecture that that the minimal number of obstacles in a regular n-gon is bounded at least linearly from below.

What is the situation for billiards with a general piece-wise smooth boundary?

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